Class-XII DETERMINANTS

A. Determinats

--To every square matrix A of order n, we can associate a number (real or complex) called determinant of the square matrix A. This may be thought of as a function which gives a unique number. It is denoted by | A | or det A or Δ. Only square matrices have determinants.

If $A = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$ determinant is $|A| = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}$

Determinant of a matrix of order one: Let A = [a] be the matrix of order 1, then determinant of A is defined to be equal to a

Determinant of a matrix of order two: Let A be the matrix of order 2 x 2, then determinant of A is defined as $a_{11} a_{22} - a_{21} a_{12}$.

Determinant of a matrix of order three: It can be determined by expanding it in terms of second order determinants. There are six ways of expanding a determinant corresponding to each of three rows and three columns. Expanding a determinant along any row or column gives same value.

--For easier calculations, we expand along that row or column which contains maximum number of zeros.

--While expanding, instead of multiplying by $(-1)^{i+j}$, we can multiply by +1 or –1 as $(i + j)$ is even or odd.

--If A = kB where A and B are square matrices of order n, then $|A| = k^n |B|$

B. Properties of Determinants

--These simplifies evaluation by obtaining maximum number of zeros in a row or a column.

1. The value of the determinant remains unchanged if its rows and columns are interchanged.

-- Thus, | A | = det | A′ |, where A′ = transpose of A.

2. If we interchange any two rows (or columns), then sign of determinant changes.

3. If any two rows (or columns) are identical or proportional, then value of determinant is zero.

4. If we multiply each element of a row (or column) by constant k, then value gets multiplied by k.

--By this property, we can take out any common factor from any one row or any one column.

Property-5: If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

Property-6: If, to each element of any row or column of a determinant, the equi-multiples of corresponding elements of other row (or column) are added, then value of determinant remains the same

--If Δ_1 is the determinant obtained by applying $R_i \rightarrow kR_j$ or $C_i \rightarrow kC_j$ to the determinant Δ, then $\Delta_1 = k\Delta$.

C. Area of a Triangle

--The area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3), is given by the expression $1/2\ [x_1(y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)]$. This expression can be written in the form of a determinant as

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

--Since area is a positive quantity, we always take the absolute value of the determinant.

--If area is given, use both positive and negative values of the determinant for calculation.

--Area of triangle formed by 3 collinear points is zero.

D. Minors and Cofactors

Minor of an element a_{ij} of a determinant A is the determinant obtained by deleting its ith row and jth column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

--Minor of an element of a determinant of order n *where* $(n \geq 2)$ is a determinant of order $n - 1$.

Cofactor of an element a_{ij}, denoted by A_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is minor of a_{ij}.

--If elements of a row (or column) are multiplied with cofactors of any other row (or column), their sum is zero.

--Determinant Δ = sum of the product of elements of any row (or column) with their corresponding cofactors.

E. Adjoint and Inverse of a Matrix

The adjoint of a square matrix $A = [a_{ij}]_{nxn}$ is defined as the transpose of the matrix $[A_{ij}]_{nxn}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by *adj* A.

--For a square matrix of order two, the *adj* A can also be obtained by interchanging a_{11} and a_{22} and by changing signs of a_{12} and a_{21},

Theorems:

1. If A be any given square matrix of order n, then A(*adj*A) = (*adj*A) A= AI, where I is identity matrix of order n.

--A square matrix A is said to be singular if |A| = 0.

--A square matrix A is said to be non-singular if |A| ≠ 0.

2. If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.
3. The determinant of the product of matrices is equal to product of their respective determinants, that is, |AB| = |A| |B|, where A and B are square matrices of same order
4. A square matrix A is invertible if and only if A is non-singular matrix.

F. Application of Determinants and Matrices

Solution of system of linear equations using inverse:

If $a_1 x + b_1 y + c_1 z = d_1$
$a_2 x + b_2 y + c_2 z = d_2$
$a_3 x + b_3 y + c_3 z = d_3$

Then, these equations can be written as, AX = B, i.e.,

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

A system of equation is consistent or inconsistent according to its solution exists or not.

Case-I: If |A| ≠ 0, the system has unique solution.

Case-II: If |A| = 0, we calculate (*adj* A) B.

If (*adj* A) B ≠ O, Inconsistent, no solution.

If (*adj* A) B = O, Either consistent or inconsistent, either infinitely many solutions or no solution.