Class-XII INTEGRALS (Part-2/2)

Contd from… (1/2)

Integrals of some specific types

a. $\int \frac{dx}{ax^2+bx+c}$ and $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ reduce these to specific form and obtain integral using standard formulae.

b. $\int \frac{(px+q)dx}{ax^2+bx+c}$ and $\int \frac{(px+q)dx}{\sqrt{ax^2+bx+c}}$ To determine real numbers A and B, we equate from both sides the coefficients of x and the constant term.

D. Integration by Partial Fractions

-- Recall that a rational function is defined as the ratio of two polynomials in the form P(x)/ Q(x), where P (x) and Q(x) are polynomials in x and Q(x) ≠ 0.

-- If the degree of P(x) is less than the degree of Q(x), then the rational function is called proper, otherwise, it is called improper. The improper rational functions can be factorised and reduced to the proper rational factors. The integrand is written as a sum of simpler rational functions as the sum of partial fractions of following types:

1. $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$, a ≠ b
2. $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$
3. $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4. $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5. $\frac{px^2+qx+r}{(x-a)\,(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx + C}{x^2+bx+c}$

D. Integration by Parts

-- The integral of the product of two functions = (first function) × (integral of the second function) – Integral of [(differential coefficient of first function) × (integral of second function). For given $f(x)$ and $f(g)$ we have:

$\int f(x)\, g(x)\, dx = f(x) \int g(x)\, dx - \int [\frac{d}{dx} f(x). \int g(x)\, dx]\, dx$

-- Care must be taken in choice of the first function and the second function. Obviously we must take that function as second function whose integral is well known to us.

Integrals of some specific types

1. $\int e^x [f(x) + f'(x)]\, dx = \int e^x f(x)\, dx = C$
2. $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2-a^2} \right| + C$
3. $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2+a^2} \right| + C$
4. $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

E. Definite Integrals

-- A definite integral is denoted by $\int_a^b f(x)\, dx$, where a is called the lower limit of the integral and b is called the upper limit of the integral.

Definite Integral as the limit of a sum

$\int_a^b f(x)\, dx$

$= (b - a) \lim_{h \to 0} h\, [\, f(a) + f(a + h) + ... + f(a + (n-1)\, h\,]$

where $h = \frac{b-a}{n} \to 0$ as $n \to \infty$

D. Fundamental Theorem of calculus

Definite Integral as Area function

The area of the region bounded by the curve $y = f(x)$, is defined by $\int_a^b f(x)\, dx$ the ordinates $x = a$, $x = b$ and x-axis.

$A(x) = \int_a^b f(x)\, dx$

Theorem 1st: Let f be a continuous function on the closed interval [a, b] and let A (x) be the area function.
Then $A'(x) = f(x)$, for all $x \in [a, b]$.

Theorem 2nd: Let f be continuous function defined on the closed interval [a, b] and F be an anti-derivative of f.
Then $\int_a^b f(x)\, dx = [F(x)] = F(b) - F(a)$.

In other words, $\int_a^b f(x)\, dx$ = (value of the anti-derivative F of f at the upper limit b – value of the same anti derivative at the lower limit a).

E. Evaluation of Definite Integrals by Substitution

-- To evaluate $\int_a^b f(x)\, dx$, by substitution, the steps could be as follows:

1. Consider the integral without limits and substitute, $y = f(x)$ or $x = g(y)$ to reduce the integral to a known form.
2. Integrate the new integrand with respect to new variable without mentioning the constant.
3. Resubstitute for the new variable and write the answer in terms of the original variable.
4. Find the values of answers obtained in (3) at the given limits of integral and find the difference of the values at the upper and lower limits.

F. Some Properties of Definite Integrals

Property-1: $\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$

Property-2: $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$ and $\int_a^a f(x)\, dx = 0$

Property-3: $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$

Property-4: $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

Property-5: $\int_0^b f(x)\, dx = \int_0^b f(a-x)\, dx$

Property-6: $\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a-x)\, dx$

Property-7: $\int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx$, if $f(2a-x) = f(x)$ and 0 if $f(2a-x) = -f(x)$

Property-8: (i) $\int_{-a}^a f(x)\, dx = 2\int_0^a f(x)\, dx$, if f is an even function, i.e., if $f(-x) = f(x)$.
(ii) $\int_a^b f(x)\, dx = 0$, if f is an odd function, i.e., if $f(-x) = -f(x)$.